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DN-003-1164002 Seat No. ________

M. Sc. (Sem. IV) (CBCS) Examination

March - 2022

Mathematics : CMT - 4002

(Integration Theory)

Faculty Code : 003

Subject Code : 1164002

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) Each question carries **14** marks.
(2) There are **5** questions in total.

1 Answer the following questions. **14**

(A) Define : Counting measure. Also find the counting measure of a set $A = \{\mathbb{R}^{-} \cup [0, 2022]\} \cap \mathbb{N}$

(B) Give the example of measure zero set need not be a null set with required justification.

(C) Give only statement of Jordan Decomposition Theorem.

(D) Prove that, every measurable subset of a negative set is negative.

(E) Let X be a locally compact T_2 – space and K be a compact G_δ – set in then prove that, $K \in B_a(X)$.

(F) If μ^* is an outer measure on a set X and $\beta \in P(X)$ be such that $\mu^* \beta = 0$ then prove that, β is μ^* measurable.

(G) Prove that, every compact subset of K of a Hausdorff space is closed.

2 Answer ***any two*** questions : **14**

(A) Let μ be a signed measure on $(X, \mathcal{A})$. Then prove that $\exists$ a positive set A with respect to μ and a negative set B with respect to μ such that $X = A \cup B, A \cap B = \phi$.

(B) Let γ be a signed measure on $(X, A), (\gamma^+, \gamma^-)$ be Jordan decomposition of γ. Prove that, for $E \in A$.

(a) E is a positive set with respect to γ if and only if $\gamma^- (E) = 0$.

(b) E is a negative set with respect to γ if and only if $\gamma^+ (E) = 0$.

(c) E is a null set with respect to γ then $|\gamma| (E) = 0$.

(C) State and prove : Lebesgue Decomposition theorem.

3 Answer the following questions : **14**

(A) If μ_1, μ_2 are two measure on a measurable space (X, A) and at least one of them is finite then prove that, $\mu_1 - \mu_2$ is a signed measure on (X, A).

(B) If μ^* is an outer measure on a set X and $B = \{E \subseteq X / E \text{ is } \mu^* - \text{measurable}\}$. Prove that, B is $\sigma -$ algebra of subset of X.

OR

3 Answer the following questions : **14**

(A) If X is a countable set and is the counting measure on *(X, P(X))*. Prove that, $L^p(\mu) \cong l^p, 1 \le p \le \infty$.

(B) Define : Measure absolutely continuous with respect to another measure and mutually singular measures. If (X, A) is a measurable space and γ, μ are signed measures on $(X, A), \gamma \perp \mu \ll \mu$ then prove that, $\gamma = 0$.

4 Answer ***any two*** questions : **14**

(A) State and prove : Radon - Nikodym theorem for measure.

(B) Let $\mathcal{C}$ be a semi algebra of subset of a set S and $\mu : \mathcal{C} \to [0, \infty]$ be such that

(a) $c \in C, = \bigcup_{i=1}^{n} c_i$. prove that, $\eta(c) = \sum_{i=1}^{n} \mu(c_i), \forall n \in \mathbb{N}, c_i \in C$ and $c_i \cap c_j = \phi, \forall i, j$

(b) $c \in C, c = \bigcup_{n=1}^{\infty} c_n$. prove that, $\mu(c) = \sum_{i=1}^{\infty} \mu(c_n), \forall c_i \in C$ and $c_i \cap c_j = \phi, \forall i, j$.

(C) Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \gamma)$ be $\sigma -$finite complete measure space, R be the semi algebra of all measurable rectangles in $X \times Y$ and $E \in R_{\sigma\delta}$ such that $(\mu \, X \, \gamma) \, (E) < \infty$. Prove that, a function $g : X \to [0, \infty]$ defined by $g(x) = \gamma(E_x), \forall x \in X$ is measurable and $\int_x g \, d\mu = (\mu \times \gamma)(E)$.

5 Answer ***any two*** questions : **14**

(1) Let X be a topological space. Prove that,

(a) For $F \subseteq X, X_F : X \to \{0,1\}$ is upper semi continuous if and only if F is closed in X.

(b) If $f_a : X \to \{0,1\}$ are upper semi continuous, $\forall a \in \wedge$ then in $f_{a \in \wedge} f_a$ is also upper semi continuous on X.

(2) Let m be the σ-algebra generated by all legesgue measurable subsets of $\mathbb{R}$ and μ be the lebesgue measure on $(\mathbb{R}, m)$. Prove that, μ is regular.

(3) Let X be a locally compact spareable metric space. Prove That, $B_0(X) = B_a(X)$.

(4) Let X be a locally compact T_2-space. Prove that, $B_a(X)$ is the σ-algebra generated by all compact G_δ-sets in X.